**The Cauchy-Schwarz Inequality**

Together with AM-GM, Schur's inequality, Jensen's inequality or Hölder's inequality, this is a fundamental result, with remarkable applications. Trying to present some of the faces of this inequality, we will insist on the diversity of the problems that can be solved using it. The main question is: how do we recognize an inequality that can be solved using this method? It is very hard to say this clearly, but it is definitely good to think of Cauchy-Schwarz whenever we have sums of radicals or sums of squares and especially when we have expressions involving fractions.

First, let us speak about some problems in which it is better to apply the direct form of the Cauchy-Schwarz inequality:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

The main difficulty is to choose $a_i$ and $b_i$. We will see that in some cases this is trivial, while in some other cases it is very difficult. Let us solve some problems now:

**Example 1**

Let $x_1, x_2, \ldots, x_{n+1} > 0$ such that $x_1 + x_2 + \ldots x_n = x_{n+1}$. Prove that:

$$\sum_{i=1}^{n} \sqrt{x_i(x_{n+1} - x_i)} \leq \sqrt{\sum_{i=1}^{n} x_{n+1}(x_{n+1} - x_i)}.$$

Romanian IMO Team Selection Test, 1996

**Solution:**

Even a blind man can see here that we must apply Cauchy-Schwarz in the following form:

$$\left(\sum_{i=1}^{n} \sqrt{x_i(x_{n+1} - x_i)}\right)^2 \leq \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} (x_{n+1} - x_i)\right) = \sum_{i=1}^{n} x_{n+1}(x_{n+1} - x_i).$$

After this easy example, we will discuss a much more difficult problem:

**Example 2**

Prove that if $x, y, z$ are reals such that $x^2 + y^2 + z^2 = 2$, then the following inequality holds:

$$x + y + z \leq 2 + xyz.$$

IMO Shortlist, proposed by Poland

**Solution:**

Why should we think about Cauchy-Schwarz inequality? The reason is clear: the relation we are asked to prove can be written as $x(1-yz)+y+z \leq 2$ and we have a bound for a sum of squares: $x^2+y^2+z^2$. Anyway, there are lots of ways to apply Cauchy-Schwarz. The choise $(x(1-yz)+y+z)^2 \leq (x^2+y^2+z^2)(2+(1-yz)^2)$ is not very happy. So, maybe it is better to look to $y+z$ as to a single number. If we also observe that we have equality when $x=1$, $y=1$ and $z=0$ (for example), the choice $x(1-yz)+y+z \leq \sqrt{(x^2+(y+z)^2)(1+(1-yz)^2)}$ becomes natural. So, we must prove that $2(1+yz)(2-2yz+y^2z^2) \leq 4 \Leftrightarrow y^3z^3 \leq y^2z^2$, which is easy, since $2 \geq y^2+z^2 \geq 2yz$.

Another highly non-trivial application of the inequality in the title is the following problem:

**Example 3**

Let $a,b,c,x,y,z$ be positive reals such that $ax+by+cz=xyz$. Prove that:

$$x+y+z > \sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}.$$

Dung Tran Nam

**Solution:**

This time the presence of the sum of radicals in the RHS gives us the sign that Cauchy might work. It would be nice if we had $(x+y+z)^2 > 6(a+b+c)$, but it is not the case. It is also clear that $xyz$ does not play an important role anywhere in the problem, so it should disappear. Thus, we write $\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}=1$ and now the substitution $a=yzu$, $b=zxu$ and $c=xyw$ becomes natural.

So, we must prove that $\sqrt{z(yu+xv)}+\sqrt{x(zv+yw)}+\sqrt{y(zu+xw)} < x+y+z$ for $u+v+w=1$. Now, one can see the form of the Cauchy-Schwarz inequality: $\left(\sqrt{z(yu+xv)}+\sqrt{x(zv+yw)}+\sqrt{y(zu+xw)}\right)^2 \leq (x+y+z)(yu+yw+xv+xw+zu+zv)$ and the latter is of course smaller than $(x+y+z)^2$ since $u+v+w=1$.

We saw that we can apply Cauchy-Schwarz when we have sums. What about products? The following example will show that we need some imagination in this case:

**Example 4**

Let $n>1$ be an integer and $a_1,\ldots,a_n$ be positive reals. Prove the inequality:

$$(a_1^3+1)\cdots(a_n^3+1) \geq (a_1^2a_2+1)\cdots(a_n^2a_1+1).$$

Czech-Slovak-Polish Match, 2001

**Solution:**

We try to apply Cauchy-Schwarz for each factor of the product in the RHS. It is natural to write $(1+a_1^2a_2)^2 \leq (1+a_1^3)(1+a_2^2a_1)$, since we need $1+a_1^3$, which appears in

the LHS. Similarly, we can write $(1+a_2^2a_3)^2 \le (1+a_2^3)(1+a_3^2a_2), \dots, (1+a_n^2a_1)^2 \le (1+a_n^3)(1+a_1^2a_n)$. Multiplying we obtain $\left((a_1^2a_2+1)\cdots(a_n^2a_1+1)\right)^2 \le (a_1^3+1)\cdots(a_n^3+1)(1+a_2^2a_1)\cdots(1+a_1^2a_n)$ $(*)$. Well, it seems that Cauchy does not work for this one. False! We use again the same argument to find that $\left((1+a_2^2a_1)\cdots(1+a_1^2a_n)\right)^2 \le (a_1^3+1)\cdots(a_n^3+1)(a_1^2a_2+1)\cdots(a_n^2a_1+1)$ $(**)$. Thus, if $(a_1^2a_2+1)\cdots(a_n^2a_1+1) \ge (1+a_2^2a_1)\cdots(1+a_1^2a_n)$, $(*)$ saves us, otherwise $(**)$ finishes the solution.

Is it time now to solve some harder problems. We will need luck and especially will to solve them:

**Example 5**

Let $x, y > 0$ be such that $x^2+y^3 \ge x^3+y^4$. Prove that $x^3+y^3 \le 2$.

Rusia, 1999

**Solution:**

The idea is to majorize $x^3+y^3$ with $A(x^3+y^4)$ for a certain $A$, which seems reasonable, looking at the exponents. So, we can try some tricks with Cauchy-Schwarz and AM-GM: $(x^3+y^3)^2 \le (x^3+y^4)(x^3+y^2) \le (x^2+y^3)(x^3+y^2) \le \left(\frac{x^2+y^2+x^3+y^3}{2}\right)^2$. Thus we have establish that $x^3+y^3 \le x^2+y^2$. But $(x^2+y^2)^2 \le (x+y)(x^3+y^3)$ and so $x^2+y^2 \le x+y \le \sqrt{2(x^2+y^2)}$ from where we find that $x^2+y^2 \le 2$ and consequently $x^3+y^3 \le 2$.

Another problem for which looking at the exponents is the key of the solution is the following:

**Example 6**

Prove that if $a, b, c > 0$ verify $a+b+c = 1$ then we have the inequality:

$$(ab)^{\frac{5}{4}} + (bc)^{\frac{5}{4}} + (ca)^{\frac{5}{4}} < \frac{1}{4}.$$

Dinu Teodorescu, proposed for ONM, 2002

**Solution:**

The exponents are so ugly, that it is clear we should make them more nice. Cauchy-Schwarz is the best therapy, since we have:

$$\left((ab)^{\frac{5}{4}} + (bc)^{\frac{5}{4}} + (ca)^{\frac{5}{4}}\right)^2 \le (\sqrt{ab}+\sqrt{bc}+\sqrt{ac})\left((ab)^2+(bc)^2+(ca)^2\right).$$

It is impossible not to see that $\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \le 1$, so it would be enough to prove that $\sum (ab)^2 < \frac{1}{16}$. This is not a very difficult task, especially because we do not have radicals. But Cauchy cannot be used anymore. Let us suppose that $a \ge b \ge c$.

Then $\frac{1}{16}$ must come from something like $x^2y^2$ with $x+y=1$. Looking again at the ordering, the best choice is:

$$\frac{1}{16} \geq a^2(b+c)^2 = (ab)^2 + a^2bc + (ca)^2 + a^2bc > \sum (ab)^2.$$

**Example 7**

Let $x_1, \ldots, x_n,\ y_1, \ldots, y_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = 1$. Prove that:

$$(x_1y_2 - x_2y_1)^2 \leq 2\left|1 - \sum_{i=1}^{n} x_iy_i\right|.$$

Korea, 2001

**Solution:**

This problem seems to be made to apply the Cauchy-Schwarz inequality. Unfortunately, with a direct use of the formula, we just can find that $\left|\sum_{i=1}^{n} x_iy_i\right| \leq 1$. We will see that this will help us a lot. What can we do with $(x_1y_2 - x_2y_1)^2$? If we know one of the proofs of the Cauchy-Schwarz inequality, Lagrange's identity, then we could try a brutal estimation:

$(x_1y_2 - x_2y_1)^2 \leq \sum_{1 \leq i < j \leq n} (x_iy_j - x_jy_i)^2 = \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) - \left(\sum_{i=1}^{n} x_iy_i\right)^2 = \left(1 - \sum_{i=1}^{n} x_iy_i\right)\left(1 + \sum_{i=1}^{n} x_iy_i\right)$. Fortunately, the estimations works since $1 + \sum_{i=1}^{n} x_iy_i \leq 2$. Thus, we have a two lines solution for this problem.

A beautiful application of Cauchy-Schwarz inequality is the following:

**Example 8**

Let $a, b, c, x, y, z$ be reals such that $(a+b+c)(x+y+z) = 3$ and $(a^2+b^2+c^2)(x^2+y^2+z^2) = 4$. Prove that $ax+by+cz \geq 0$.

Vasile Cîrtoaje

**Solution:**

Again, this seems a direct application of the method, but we will see again that we will have troubles in applying it. How can we apply Cauchy-Schwarz here? The answer will follow from another question: what does $ax+by+cz \geq 0$ mean? It is equivalent to $(ax+by+cz-t)^2 \leq (ax+by+cz)^2 + t^2$ for a certain positive real $t$. Thus, it would be nice to find such a $t$. This leads to the evaluation of the expression $ax+by+cz-t$. Trying to homogenize everything, we have the following natural chain of relations

and inequalities: $|ax+by+cz-t|^2 = \left(ax+by+cz-\frac{t}{3}(a+b+c)(x+y+z)\right)^2 = \left(\sum a\left(\frac{t}{3}(x+y+z)-x\right)\right)^2 \leq \left(\sum a^2\right)\cdot\left(\sum\left(\frac{t}{3}(x+y+z)-x\right)^2\right)$. So, we must look for a $t$ with the property that $\frac{4}{x^2+y^2+z^2}\cdot\sum\left(\frac{t}{3}(x+t+z)-x\right)^2 \leq t^2$ $(*)$. In this way, we will have $(ax+by+cz-t)^2 \leq (ax+by+cz)^2+t^2$ and the solution would end. But $(*)$ is equivalent to $\frac{t(t-2)}{3}(x+y+z)^2+x^2+y^2+z^2 \leq \frac{t^2}{4}(x^2+y^2+z^2)$. Clearly, the best choice is $t=2$. So, indirectly, we have found that $|2-ax-by-cz| \leq 2$ and it follows that $ax+by+cz \geq 0$.

We have not seen radicals for a long time, so it is time to solve such a problem too:

**Example 9**

Prove that if $x,y,z \in [-1,1]$ verify $x+y+z+xyz=0$, then we have:

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1} \leq 3.$$

Gabriel Dospinescu

**Solution:**

We have radicals, so we first try Cauchy in the obvious form:

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1} \leq \sqrt{3(x+y+z+3)}.$$

But is this smaller than 3? Well, if $x+y+z \leq 0$, it is. Let us suppose it is not the case. Thus, $xyz < 0$. Let $z < 0$. It follows that $x,y \in (0,1]$. We do not give up and try to use again Cauchy-Schwarz, but for the first two radicals:

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1} \leq \sqrt{2x+2y+4}+\sqrt{z+1}.$$

And we have to prove that: $\sqrt{2x+2y+4}+\sqrt{z+1} \leq 3 \Leftrightarrow \frac{2(x+y)}{2+\sqrt{2x+2y+4}} \leq \frac{-z}{1+\sqrt{z+1}} \Leftrightarrow \frac{-z(xy+1)}{2+\sqrt{2x+2y+4}} \leq \frac{-z}{1+\sqrt{z+1}} \Leftrightarrow 2xy+2(1+xy)\sqrt{1+z} \leq \sqrt{2x+2y+4}$. Since $1+z = \frac{(1-x)(1-y)}{1+xy}$, everything comes down to proving that $xy+\sqrt{(1-x)(1-y)(1+xy)} \leq \sqrt{1+\frac{x+y}{2}}$. We use Cauchy so that $1-x$ vanishes from the LHS. That is why we try to use it in the following form: $xy+\sqrt{(1-x)(1-y)(1+xy)} = \sqrt{x}\cdot\sqrt{xy^2}+\sqrt{1-x}\cdot\sqrt{1+xy-y-xy^2} \leq \sqrt{1+xy-y} \leq 1 \leq \sqrt{1+\frac{x+y}{2}}$. And the problem is solved.

Maybe the hardest example of them all is the following problem:

**Example 10**

Prove that for all $a, b, c, x, y, z > 0$ we have the following inequality:

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(z+x) + \frac{c}{a+b}(x+y) \geq 3 \cdot \frac{xy+yz+zx}{x+y+z}.$$

Walther Janous, Crux Mathematicorum

Here, no sign of Cauchy can be seen. Yet, there are two different solutions of this problem using Cauchy-Schwarz:

**Solution 1:**

$\sum \frac{a}{b+c}(y+z) + \sum(y+z) = \left(\sum a\right)\left(\frac{y+z}{b+c}\right) = \frac{1}{2}\left(\sum(b+c)\right)\left(\sum \frac{y+z}{b+c}\right) \geq \frac{1}{2}\left(\sum \sqrt{y+z}\right)^2$. We will show that:

$$\frac{1}{2}\left(\sum(\sqrt{y+z})\right)^2 \geq \frac{3\sum yz}{\sum x} + 2\sum x \qquad (1)$$

from which our result follows. (1) is equivalent to

$$\left(\sum(x+\sqrt{x^2+xy+yz+zx})\right)\left(\sum x\right) \geq 3\left(\sum yz\right) + 2\left(\sum x\right)^2.$$

Since

$$\sum \sqrt{x^2+(xy+yx+zx)} \geq \sqrt{\left(\sum x\right)^2 + 9\left(\sum yz\right)}.$$

It is enough to show that

$$\left(\sum x\right)\sqrt{\left(\sum x\right)^2 + 9\left(\sum yz\right)} \geq \left(\sum x\right)^2 + 3\left(\sum yz\right)$$

which after squaring both sides becomes a trivial inequality.

**Solution 2:**

Due to the homogeneity in $x, y, z$ we can suppose that $x+y+z = 1$. The inequality becomes:

$$3(xy+yz+zx) + \frac{a}{b+c}x + \frac{b}{c+a}y + \frac{c}{a+b}z \leq \sum \frac{a}{b+c}.$$

Now we can start applying Cauchy-Schwarz. The most natural is to start with $\frac{a}{b+c}x + \frac{b}{c+a}y + \frac{c}{a+b}z \leq \sqrt{x^2+y^2+z^2} \cdot \sqrt{\sum\left(\frac{a}{b+c}\right)^2}$. Now we try to use $3(xy+yz+zx)$ so that we can add (using Cauchy-Schwarz) $2(xy+yz+zx)$ to $x^2+y^2+z^2$. So, we need something like $\sqrt{\sum xy} \cdot \sqrt{A} + \sqrt{\sum xy} \cdot \sqrt{B} + \sqrt{\sum x^2} \cdot \sqrt{\sum\left(\frac{a}{b+c}\right)^2} \leq \sqrt{\sum x^2 + 2\sum xy} \cdot \sqrt{A+B+\sum\left(\frac{a}{b+c}\right)^2}$.

We must also have $\sqrt{A}+\sqrt{B}=3\sqrt{xy+yz+zx}$ and $A+B$ should be as small as possible. Thus, the only choice is $A=B=\frac{9}{4}(xy+yz+zx)\leq\frac{3}{4}$ and all we need is to prove that $\sqrt{2\cdot\frac{3}{4}+\sum\left(\frac{a}{b+c}\right)^2}\leq\sum\frac{a}{b+c}\Leftrightarrow\sum\frac{ab}{(c+a)(c+b)}\geq\frac{3}{4}$. But this is an easy task, since it reduces to $\sum a(b-c)^2\geq 0$.

It is now time to speak about a direct consequence of the Cauchy-Schwarz inequality, but which is so useful that it is better to know it as a distinct result. This is the inequality $\sqrt{(a+c)(b+d)}\geq\sqrt{ab}+\sqrt{cd}$, true for all $a,b,c,d$ non-negative reals. First, we will discuss two easy problems:

**Example 11**

Prove that for all $a,b,\in[0,1]$ we have the following inequality:

$$\sqrt{[a^2+(1-b)^2]\cdot[b^2+(1-a)^2]}+\sqrt{(a^2+b^2)\cdot[(1-a)^2+(1-b)^2]}\geq 1.$$

Constantin Buşe

**Solution:**

We will see that in fact the inequality is true for all reals $a,b$. Using the trick described above, we can write $\sqrt{[a^2+(1-b)^2]\cdot[b^2+(1-a)^2]}\geq|a|\cdot|b|+|1-a|\cdot|1-b|$ and $\sqrt{(a^2+b^2)\cdot[(1-a)^2+(1-b)^2]}\geq|a(1-b)|+|b(1-a)|$. So, we need only to prove that $|ab|+|1-a-b+ab|+|a-ab|+|b-ab|$ is at least 1, which is and easy task since this quantity is at least $|ab+(1-a-b+ab)+a-b-2ab|=1$.

**Example 12**

Prove that for all positive numbers $x,y,z$ we have the inequality:

$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(x+y)(z+y)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq 1.$$

Walther Janous, Crux Mathematicorum

**Solution:**

It is clear how the trick should be applied. We have $\sum\frac{x}{x+\sqrt{(x+y)(x+z)}}\leq\sum\frac{x}{2x+\sqrt{yz}}$. And now? Naturally, we will write $a=\frac{\sqrt{yz}}{x}$, $b=\frac{\sqrt{zx}}{y}$, $c=\frac{\sqrt{xy}}{z}$ and the inequality becomes $\sum\frac{1}{2+a}\leq 1$ when $abc=1$. This is easy, since it reduces after clearing the denominators to $ab+bc+ca\geq 3$.

Let us solve now two mote difficult problems:

**Example 13**

Let $a, b, c > 0$ such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$. Prove that:

$$\sqrt{a+bc} + \sqrt{b+ca} + \sqrt{c+ab} \geq \sqrt{abc} + \sqrt{a} + \sqrt{b} + \sqrt{c}.$$

APMO, 1999

**Solution:**

It is clear that we cannot apply the trick directly. Here the most important thing is to observe that from the given relation it follows that $a + bc + b + ca + a + ab = abc + a + b + c$, so we can square to get the equivalent inequality:

$$\sum \sqrt{(a+bc)(b+ca)} \geq \sum \left(\sqrt{ab} + c\sqrt{ab}\right).$$

But this is immediate since $\sqrt{(a+bc)(b+ca)} \geq \sqrt{ab} + c\sqrt{ab}$.

**Example 14**

Prove that if $x, y, z > 0$ then the following inequality holds:

$$\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} \geq 1 + \sqrt{1 + \sqrt{(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)}}.$$

Vasile Cîrtoaje

**Solution:**

Obviously, we must change a little bit the form of this terrifying inequality, so that not to have so many radicals. So, we must square. Obviously, the best way to do that is the following:

$$\left(\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} - 1\right)^2 \geq 1 + \sqrt{(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)} \Leftrightarrow$$

$\Leftrightarrow (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 2\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} +$ $\sqrt{(x^2+y^2+z^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)}$. We have the sum of two radicals in the RHS, so maybe it would be better to write $(x+y+z)\left(\frac{1}{x}+\frac{1}{z}+\frac{1}{y}\right)$ in the form $\sqrt{(a+c)(b+d)}$. Fortunately, this is easy, since $(x+y+z)\left(\frac{1}{x}+\frac{1}{z}+\frac{1}{y}\right) = \sqrt{\left(\sum x^1 + 2\sum xy\right) \cdot \left(\sum \frac{1}{x^2} + 2\sum \frac{1}{xy}\right)}$ and now the trick can be applied since:

$\sqrt{\left(\sum x^2+2\sum xy\right)\cdot\left(\sum\frac{1}{x^2}+2\sum\frac{1}{xy}\right)}\geq\sqrt{\left(\sum x^2\right)\cdot\left(\sum\frac{1}{x^2}\right)}+$
$2\sqrt{\left(\sum xy\right)\cdot\left(\sum\frac{1}{xy}\right)}$. The simple observation that $\left(\sum xy\right)\cdot\left(\sum\frac{1}{xy}\right)=\left(\sum x\right)\cdot$
$\left(\sum\frac{1}{x}\right)$ finishes the solution.

We will speak now about the most used trick in the last years in contest problems. It is a direct variant of the Cauchy-Schwarz inequality:

$$\sum_{k=1}^{n}\frac{a_k^2}{b_k}\geq\frac{\left(\sum\limits_{k=1}^{n}a_k\right)^2}{\sum\limits_{k=1}^{n}b_k},\quad\text{for all reals } a_i\text{, and positive numbers } b_i.$$

An easy application of this trick is the following variant of a problem given in the Tournament Of The Towns competition in ... (the year), which is stronger than the problem given in the contest.

**Example 15**

Prove that for all positive reals $a,b,c$ we gave the following inequality:

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ac+a^2}\geq\frac{a^2+b^2+c^2}{a+b+c}.$$

**Solution:**

It is very important to study both the exponents of the nominators of the LHS and the nominator of the RHS. If we wrote RHS as $\frac{(a^2+b^2+c^2)^2}{(a+b+c)(a^2+b^2+c^2)}$, we would know what we have to do:

$$\sum\frac{a^3}{a^2+ab+b^2}=\sum\frac{(a^2)^2}{a(b^2+ab+a^2)}\geq\frac{\left(\sum a^2\right)^2}{\sum a(a^2+ab+b^2)}.$$

So, it would be nice to have $\sum a(a^2+ab+b^2)\leq\left(\sum a\right)\cdot\left(\sum a^2\right)$, which is in fact verified with $=$.

It may seem that any inequality with fractions can be attacked in this way. It is false! There are cases when it is almost impossible to find $a_i$ and $b_i$. Let us discuss some problems in which it is not easy at all to use the trick.

**Example 16**

Let $a_1,a_2,\ldots,a_n>0$ be such that $a_1+a_2+\ldots+a_n=1$. Prove that:

$$(a_1a_2+a_2a_3+\ldots+a_na_1)\left(\frac{a_1}{a_2^2+a_2}+\ldots+\frac{a_n}{a_1^2+a_1}\right)\geq\frac{n}{n+1}.$$

**Solution:**

The main question is: from where should we obtain $a_1a_2 + a_2a_3 + \ldots + a_na_1$ ? It may appear for example from $\dfrac{a_1^2}{a_1a_2} + \ldots + \dfrac{a_n^2}{a_na_1}$ after we use the trick. So, let us try this (although it is not obvious at all that this could lead to a solution):

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \ldots + \frac{a_n}{a_1} = \frac{a_1^2}{a_1a_2} + \ldots + \frac{a_n^2}{a_na_1} \geq \frac{1}{a_1a_2 + \ldots + a_na_1}.$$

The new problem, proving that $\dfrac{a_1}{a_2^2 + a_2} + \ldots + \dfrac{a_n}{a_1^2 + a_1} \geq \dfrac{n}{n+1}\left(\dfrac{a_1}{a_2} + \ldots + \dfrac{a_n}{a_1}\right)$ seems much more difficult, but we will see that we have to make one more step to solve it. Again, we look at the RHS and try to write $\dfrac{a_1}{a_2} + \ldots + \dfrac{a_n}{a_1}$ as $\dfrac{\left(\frac{a_1}{a_2} + \ldots + \frac{a_n}{a_1}\right)^2}{\frac{a_1}{a_2} + \ldots + \frac{a_n}{a_1}}$. So, we could try the following application of this trick:

$$\frac{a_1}{a_2^2 + a_2} + \ldots + \frac{a_n}{a_1^2 + a_1} = \frac{\left(\frac{a_1}{a_2}\right)^2}{a_1 + \frac{a_1}{a_2}} + \ldots + \frac{\left(\frac{a_n}{a_1}\right)^2}{a_n + \frac{a_n}{a_1}} \geq \frac{\left(\frac{a_1}{a_2} + \ldots + \frac{a_n}{a_1}\right)^2}{1 + \frac{a_1}{a_2} + \ldots + \frac{a_n}{a_1}}.$$

We are left with an easy problem: if $t = \dfrac{a_1}{a_2} + \ldots + \dfrac{a_n}{a_1}$, then $\dfrac{t^2}{1+t} \geq \dfrac{nt}{n+1}$, or $t \geq n$. But this is immediate using AM-GM.

**Example 17**

Prove that for all $a, b, c > 0$ we have the following inequality:

$$\frac{(b+c-a)^2}{a^2 + (b+c)^2} + \frac{(c+a-b)^2}{b^2 + (a+c)^2} + \frac{(a+b-c)^2}{c^2 + (a+b)^2} \geq \frac{3}{5}.$$

Japan, 1997

**Solution:**

After an immediate fail with a direct application of the trick, one will certainly be disappointed. Yet, the form of the inequality invites us to use the trick. The most natural way would be:

$$\sum \frac{(b+c-a)^2}{a^2 + (b+c)^2} = \sum \frac{\left(\frac{b+c}{a} - 1\right)^2}{1 + \left(\frac{b+c}{a}\right)^2} \geq \frac{\left(\sum \frac{b+c}{a} - 3\right)^2}{3 + \sum \left(\frac{b+c}{a}\right)^2},$$

because in this way we obtain a nice inequality in three variables, whose properties are well-known. So, we have to show that if $x = \dfrac{b+c}{a}$, $y = \dfrac{c+a}{b}$, $z = \dfrac{a+b}{c}$, then $(x+y+z-3)^2 \geq \dfrac{3}{5}(x^2 + y^2 + z^2 + 3)$, which is equivalent to $\left(\sum x\right)^2 - 15 \sum x +$

$3\sum xy+18 \geq 0$. Unfortunately, we cannot use directly the fact that $xy+yz+zx \geq 12$. So, we should look for something like $xy+yz+zx \geq k(x+y+z)$. The best would be $k=2$ (so as to have equality when $x=y=z=2$). Indeed, after some computations this can be written as $\sum a^3 + 3abc \geq \sum ab(a+b)$, which is Schur's inequality. So, we can write:

$$\left(\sum x\right)^2 - 15\sum x + 3\sum xy + 18 \geq \left(\sum x\right)^2 - 9\sum x + 18 \geq 0,$$

the last one is obvious since $x+y+z \geq 6$.

**Example 18**

Let $a_1, \ldots, a_5 > 0$ whose sum of squares is at least 1. Prove that we have:

$$\frac{a_1^2}{a_2+a_3+a_4} + \frac{a_2^2}{a_3+a_4+a_5} + \frac{a_3^2}{a_4+a_5+a_1} + \frac{a_4^2}{a_5+a_1+a_2} + \frac{a_5^2}{a_1+a_2+a_3} \geq \frac{\sqrt{5}}{3}.$$

Mathematics and Youth

**Solution:**

This is the record of the number of times Cacuhy-Schwarz must be applied. First, it is clear that the LHS is at least $\frac{\left(\sum a_1^2\right)^2}{\sum a_1^2(a_2+a_3+a_4)}$. So, it is enough to prove that this quantity is at least $\frac{\sqrt{5}}{3}$. So, it would be better to find a bound for $\sum a_1^2(a_2+a_3+a_4)$ which depends on $\sum a_1^2$. Let us try with Cauchy-Schwarz:

$$\left(\sum \sqrt{a_1^2} \cdot \sqrt{a_1^2(a_2+a_3+a_4)^2}\right)^2 \leq \left(\sum a_1^2\right) \cdot \left(\sum a_1^2(a_2+a_3+a_4)^2\right).$$

Again we try to estimate $(a_2+a_3+a_4)^2 \leq 3(a_2^2+a_3^2+a_4^2)$ so that we can express everything in terms of $a_1^2, \ldots, a_5^2$. Now, we can substitute $x_k = a_k^2$ and we can translate everything in the following way: if $\sum x_i \geq 1$ then $\left(\sum x_1\right)^2 \geq \frac{\sqrt{5}}{3}\sqrt{\sum x_1} \cdot \sqrt{\sum 3x_1(x_2+x_3+x_4)}$, which is $\left(\sum x_1\right)^3 \geq \frac{5}{3}\sum x_1\left(x_2+x_3+x_4\right)$. Homogeneity forces us to try to prove that $\left(\sum x_1\right)^2 \geq \frac{5}{3}\sum x_1(x_2+x_3+x_4)$. But this can be written as $\sum_{k=1}^{5} x_k^2 + \sum_{k=1}^{5} x_k x_{k+1} \geq \frac{2}{5}\left(\sum_{k=1}^{5} x_k\right)^2$, or $\sum_{k=1}^{5}(x_k+x_{k+1})^2 \geq \frac{2}{5}\left(\sum_{k=1}^{5} x_k\right)^2$, which follows from another application of the Cauchy-Schwarz inequality.

But maybe the hardest example of this type is the following problem:

**Example 19**

Let $a_1, \ldots, a_n > 0$ and $n > 12$ be such that $\sum_{k=1}^{n} a_k = 1$ and $\sum_{k=1}^{n} k a_k = 2$. Prove that:

$$(a_2 - a_1)\sqrt{2} + (a_3 - a_2)\sqrt{3} + \ldots + (a_n - a_{n-1})\sqrt{n} < 0.$$

Gabriel Dospinescu

**Solution:**

How should we start, with so many conditions? The general rule is that it is better to ignore them all and to concentrate on the expression that appears in the request of the problem. Of course, $(a_2 - a_1)\sqrt{2} + (a_3 - a_2)\sqrt{3} + \ldots + (a_n - a_{n-1})\sqrt{n} = -a_1\sqrt{2} + a_2(\sqrt{2} - \sqrt{3}) + \ldots + a_{n-1}(\sqrt{n-1} - \sqrt{n}) + a_n\sqrt{n}$. So, we have to prove that $a_1 + a_2(\sqrt{3} - \sqrt{2}) + \ldots + a_{n-1}(\sqrt{n} - \sqrt{n-1}) > a_n\sqrt{n}$. To complete the expression in the LHS, let us add $-a_1$ and $a_n(\sqrt{n+1} - \sqrt{n})$. So, we have to prove that $\sum_{i=1}^{n} a_i(\sqrt{i+1} - \sqrt{i}) > a_n\sqrt{n+1} - a_1$. But $\sum_{i=1}^{n} a_i(\sqrt{i+1} - \sqrt{i}) = \sum_{i=1}^{n} \frac{a_i}{\sqrt{i} + \sqrt{i+1}}$. Since we are given that $\sum_{k=1}^{n} a_k = 1$, it is clear how we should continue: $\sum_{i=1}^{n} \frac{a_i}{\sqrt{i} + \sqrt{i+1}} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} a_i(\sqrt{i} + \sqrt{i+1})}$. Thus, we have to maximize $\sum_{i=1}^{n} a_i\sqrt{i}$ and $\sum_{i=1}^{n} a_i\sqrt{i+1}$. Looking again at the conditions of the problem, we find that Cauchy-Schwarz is again the best solution: $\sum_{i=1}^{n} a_i\sqrt{i} = \sum_{i=1}^{n} \sqrt{a_i} \cdot \sqrt{i a_i} \leq \sqrt{\sum_{i=1}^{n} a_i} \cdot \sqrt{\sum_{i=1}^{n} i a_i} = \sqrt{2}$ and similarly we find that $\sum_{i=1}^{n} a_i\sqrt{i+1} \leq \sqrt{3}$. So, we have to prove that $\sqrt{3} - \sqrt{2} > a_n\sqrt{n+1} - a_1$. If we try directly to use that $a_n \leq \frac{2}{n}$, we will fail. But we can use both conditions to find that $2 = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} (i-1)a_i > 1 + (n-1)a_i$, so that $a_n < \frac{1}{n-1}$. Thus, $a_n\sqrt{n+1} - a_1 < \frac{\sqrt{n+1}}{n-1}$. We finally see from where we have the condition $n > 12$. Because for $n > 12$ we have $\frac{\sqrt{n+1}}{n-1} < \sqrt{3} - \sqrt{2}$.

## Problems For Practice

1. Prove the following stronger inequality:

$$\frac{a}{b+c}(y+z) + \frac{b}{c+a}(x+z) + \frac{c}{a+b}(x+y) \geq \sqrt{3(xy+yz+zx)}.$$

Gabriel Dospinescu

2. Let $x, y, z > 0$ such that $x + y + z = x^3 + y^3 + z^3$. Prove that:

$$\frac{x}{x^2+1}\left(\frac{z}{y}\right)^2 + \frac{y}{y^2+1}\left(\frac{x}{z}\right)^2 + \frac{z}{z^2+1}\left(\frac{y}{x}\right)^2 \geq \frac{x+y+z}{2}.$$

Gabriel Dospinescu

3. Given $n > 1$, find the minimal value of $\dfrac{x_1^5}{x_2 + \ldots + x_n} + \ldots + \dfrac{x_n^5}{x_1 + \ldots + x_{n-1}}$, when $x_1, \ldots, x_n > 0$ have the sum of their squares 1.

Turkey, 1997

4. Let $x, y, z > 1$ such that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 2$. Prove that $\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}$.

Iran, 1998

5. Let $x, y, z > 0$ such that $xyz = 1$. Prove that:

$$\sqrt{4+9x^2} + \sqrt{4+9y^2} + \sqrt{4+9z^2} \leq \sqrt{13}(x+y+z).$$

Gabriel Dospinescu

6. Let $a, b, c > 0$ such that $abc = 1$. Prove that:

$$\frac{a}{1+b+c} + \frac{b}{1+c+a} + \frac{c}{1+a+b} \geq 1.$$

Vasile Cîrtoaje

7. Let $a, b, c \geq 0$ such that $a^2 + b^2 + c^2 \geq 2\max\{a^2, b^2, c^2\}$. Prove that:

$$(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3) \geq 4(a^6+b^6+c^6).$$

Japan, 2001

8. Let $n > 3$ and $a_1, \ldots, a_n \geq 0$, whose sum of squares is 1. Prove that:

$$\frac{a_1}{a_2^2+1} + \ldots + \frac{a_n}{a_1^2+1} \geq \frac{4}{5}\left(a_1\sqrt{a_1} + \ldots + a_n\sqrt{a_n}\right)^2.$$

Romanian Team Selection Test, 2002

9. Prove that if $a_1, \ldots, a_6 \in \left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$, then we have the following inequality:

$$\frac{a_1-a_2}{a_2+a_3} + \frac{a_2-a_3}{a_3+a_4} + \frac{a_3-a_4}{a_4+a_5} + \frac{a_4-a_5}{a_5+a_6} + \frac{a_5-a_6}{a_6+a_1} + \frac{a_6-a_1}{a_1+a_2} \geq 0.$$

Vasile Cîrtoaje

I added some problems here, you put them to adequate places.

1. Show that for $x, y, z > 0$ we have

$$6(x+y-z)(x^2+y^2+z^2)+27xyz \le 10(x^2+y^2+z^2)^{\frac{3}{2}}.$$

Dung Tran Nam

**Solution:**

Using the intuitive principle "equality holds when some two variables are equal", we can find that equality in this inequality holds for $x = y = 2z$. This information gives us the key for using Cauchy: $10(x^2+y^2+z^2)^{\frac{3}{2}} - 6(x+y-z)(x^2+y^2+z^2) = (x^2+y^2+z^2)[\frac{10}{3}(x^2+y^2+z^2)^{\frac{1}{2}}(2^2+2^2+1^2)^{\frac{1}{2}} - 6(x+y-z)] \ge (x^2+y^2+z^2)[\frac{10}{3}(2x+2y+z) - 6(x+y-z)] = (x^2+y^2+z^2)\frac{2x+2y+28z}{3}$. Thus, we are left to show that $(x^2+y^2+z^2)\frac{2x+2y+28z}{3} \ge 27xyz$. Now, it is the turn of weighted AM-GM: $(x^2+y^2+z^2)(2x+2y+28z) = \left[4\frac{x^2}{4}+4\frac{y^2}{4}+z^2\right][2x+2y+7\cdot 4z] \ge 9\left[\left(\frac{x^2}{4}\right)^4 \cdot \left(\frac{y^2}{4}\right)^4 \cdot z^2\right]^{\frac{1}{9}} \cdot 9[(2x)(2y)(4z)^7]^{\frac{1}{9}} = 81xyz$. and we are done.

(Note that this example is taken from Vietnamese MO 2002, but it is only a special case, although it is the main case, because other cases are easier. That problem was proposed by me).

2. Let $k$ be a positive real, $k \ge 8$. Show that for positive reals $a, b, c$ we have the inequality:

$$\frac{a}{\sqrt{a^2+kbc}} + \frac{b}{\sqrt{b^2+kca}} + \frac{c}{\sqrt{c^2+kab}} \ge \frac{3}{\sqrt{k+1}}.$$

Generalization of IMO 2001

**Solution:**

We have, by Cauchy:

$$\left(\sum \frac{a}{\sqrt{a^2+kbc}}\right)\left(\sum a\sqrt{a^2+kbc}\right) \ge \left(\sum a\right)^2.$$

Now, apply Cauchy again for the second sum and we have:

$$\left(\sum a\sqrt{a^2+kbc}\right)^2 = \left(\sum \sqrt{a}\sqrt{a^3+kabc}\right)^2 \le \left(\sum a\right)\left(\sum (a^3+kabc)\right).$$

All we have to do now is to show that

$$(k+1)\left(\sum a\right)^3 \ge 9\left(\sum (a^3+kabc)\right).$$

But it is equivalent to

$$(k-8)(a^3+b^3+c^3)+3(k+1)(a+b)(b+c)(c+a) \geq 27kabc$$

which is trivial by AM-GM.

3. Show that for $x \in \left[\frac{3}{2}, 5\right]$ we have

$$\left(\sqrt{2x-3}+\sqrt{15-3x}+2\sqrt{x+1}\right)^2 < 71.25.$$

CMO, stage 1, 2003

**Solution:**

$\left(\sqrt{2x-3}+\sqrt{15-3x}+2\sqrt{x+1}\right)^2 \leq \left(\sqrt{2}\sqrt{x-\frac{3}{2}}+\sqrt{\frac{3}{2}}\sqrt{10-2x}+2\sqrt{x+1}\right)^2 \leq$
$\left(2+\frac{3}{2}+4\right)\left(x-\frac{3}{2}+10-2x+x+1\right) = 71.25.$

It is easy to see that the equality cannot hold, so we have a strict inequality.

Note that the maximum of $A^2$ is 71.2496995 (computed by Maple) so our result is quite strong. In the original variant the constant 71.25 was replaced by 76 which leaves more freedom to use Cauchy.

4. Prove that if $a, b, c$ are the sides of the triangle $ABC$ then we have

$$\frac{4}{3} < \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} < \frac{5}{3}.$$

Folklore

5. Let $x, y, z$ be positive reals satisfying the following conditions:

i) $\frac{1}{\sqrt{2}} \leq z \leq \min\{x\sqrt{2}, y\sqrt{3}\}$;

ii) $x+z\sqrt{3} \geq \sqrt{6}$;

iii) $y\sqrt{3}+z\sqrt{6} \geq \sqrt{20}$.

Find the maximal value of $P(x,y,z) = \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$.

**Solution:**

From the first condition we have:

$$\sqrt{2} \geq z > \frac{\sqrt{2}}{x} \Rightarrow \frac{1}{z^2} \leq 2, \frac{z^2}{x^2} < \frac{1}{2}.$$

From the second condition, applying Cauchy, we have:

$$x^2+3z^2 \geq 3 \Rightarrow \frac{2}{3}+\frac{2z^2}{x^2} \geq \frac{2}{x^2}.$$

Thus, $\frac{1}{x^2}+\frac{1}{z^2}=\frac{2}{x^2}+\frac{1}{z^2}-\frac{1}{x^2}\le\frac{2}{3}+\frac{2z^2}{x^2}+\frac{1}{z^2}\left(1-\frac{z^2}{x^2}\right)\le\frac{2}{3}+\frac{2z^2}{x^2}+2\left(1-\frac{z^2}{x^2}\right)=\frac{8}{3}$ (1). Similarly, from the first and third conditions, we can prove that $\frac{1}{y^2}+\frac{1}{z^2}\le\frac{13}{5}$ (2).

From (1) and (2) we get that

$$P(x,y,z)=\frac{1}{x^2}+\frac{1}{z^2}+\frac{2}{\frac{1}{y^2}+\frac{1}{z^2}}\le\frac{8}{3}+\frac{26}{5}=\frac{118}{15}.$$

Equality holds iff $x=\sqrt{\frac{3}{2}}$, $y=\sqrt{\frac{5}{3}}$, $z=\sqrt{\frac{1}{2}}$.

6. Let $x\in[0,1]$. Show that

$$x(13\sqrt{1-x^2}+9\sqrt{1+x^2})\le 16.$$

Dung Tran Nam, Olympiad on 30 April, 1996

**Solution:**

Of course we can use calculus to solve this problem, but following solution is nice and short: $x(13\sqrt{1-x^2}+9\sqrt{1+x^2})=\sqrt{13x^2}\cdot\sqrt{13(1-x^2)}+\sqrt{27x^2}\cdot\sqrt{3(1+x^2)}\le\sqrt{13x^2+27x^2}\cdot\sqrt{13(1-x^2)+3(1+x^2)}=2\sqrt{10x^2(16-10x^2)}\le 2\frac{10x^2-16-10x^2}{2}=16.$

7. Prove that for arbitrary $2n$ real numbers $a_1,a_2,\dots,a_n,x_1,x_2,\dots,x_n$ we have

$$\sum a_ix_i+\sqrt{\left(\sum a_i^2\right)\left(\sum x_i^2\right)}\ge\frac{2}{9}\left(\sum a_i\right)\left(\sum x_i\right).$$

When does equality hold?

Kvant 1989, Mathematics and Youth 1997

(proposed by me for Mathematics and Youth. Kvant variant was for $n=3$)

8. Prove that for any positive reals $a,b,c$ we have

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge(ab+bc+ca)^3.$$

Unknown source

**Solution:**

Apply generalized Cauchy:

$$(x_1+x_2+x_3)(y_1+y_2+y_3)(z_1+z_2+z_3)\ge(\sqrt[3]{x_1y_1z_1}+\sqrt[3]{x_2y_2z_2}+\sqrt[3]{x_3y_3z_3})^3$$

for $x_1 = a^2$, $x_2 = ab$, $x_3 = b^2$, $y_1 = c^2$, $y_2 = b^2$, $y_3 = bc$, $z_1 = ac$, $z_2 = a^2$, $z_3 = c^2$ we get the desired result.

9. Prove that for $a, b, c > 0$ we have

$$\sqrt{\frac{ab + bc + ca}{3}} \leq \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}}.$$

Carson's inequality

10. For any positive reals $a, b, c$ prove that:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}.$$

Vietnamese Team Selection 1982